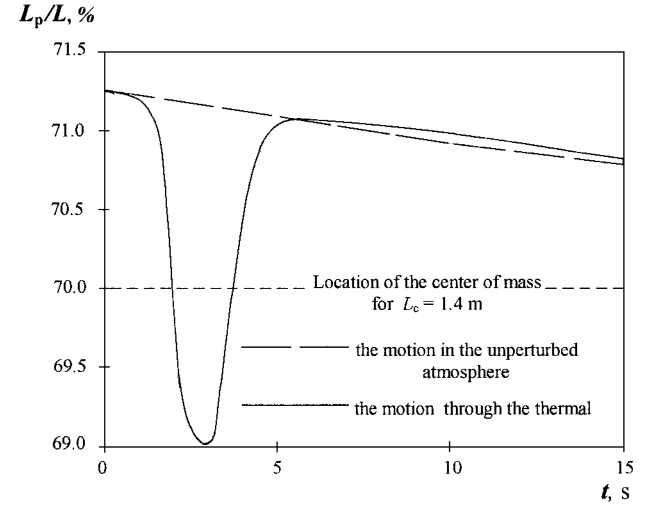

Fig. 3 Distance between the center of pressure L_p and the vertex of the cone vs flight time t.

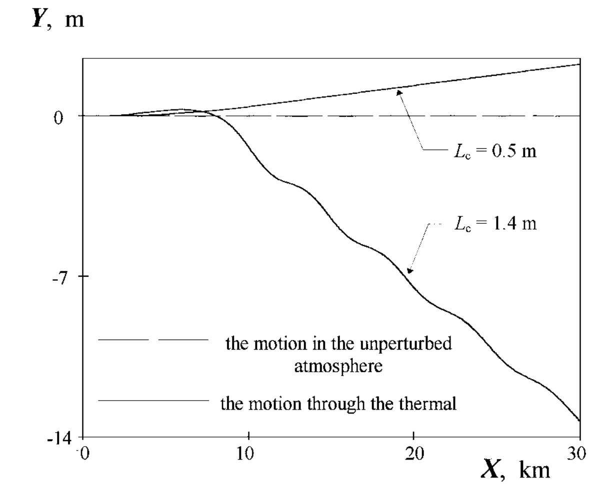

Fig. 4 Projections of body trajectories onto the horizontal plane (lateral drift Y vs X coordinate); X and Y axes are the same as in Fig. 1.

When $L_c = 0.5$ m, the presence of the thermal along the path of the body has practically no effect on the trajectory; when $L_c = 1.4$ m, the change of the trajectory is much more significant than in the first case due to less stability of the flight (or its instability in the thermal domain).

The great difference between flight stability in cases $L_c = 0.5$ and 1.4 m causes an interesting effect obtained from the computations. Projections of the body trajectory onto the horizontal plane are shown in Fig. 4. At each point along the flight trajectory into the thermal domain, the velocity $\hat{W}$ has a positive Y component. On entering the thermal domain, the body's center of mass begins to drift to the left under the influence of an incident flow, and the body's axis of symmetry begins to turn in the direction of the incident flow (clockwise in Fig. 4). When $L_c = 0.5$ m, the body lines up rapidly with an incident flow and continues to shift to the left. When $L_c = 1.4$ m, it turns out that clockwise rotating body gets into the region where it loses stability (see Fig. 3). In this situation, the clockwise rotation of the body is enhanced even more. As a result, on leaving the thermal the body goes to the right, i.e., in the direction in which it has been turned. The trajectory oscillations in Fig. 4 are due to the yaw angle oscillations.

References

[1] Muzafarov, I. F., Tirskiy, G. A., Utyuzhnikov, S. V., and Yamaleev, N. K., "Numerical Simulation of the Flow over a Body Flying Through a Thermal in a Stratified Atmosphere," *Computers and Fluids*, Vol. 23, No. 2, 1994, pp. 295–304.

[2] Tirskiy, G. A., Utyuzhnikov, S. V., and Yamaleev, N. K., "Efficient Numerical Method for Simulation of Supersonic Viscous Flow Past a Blunted Body at Small Angle of Attack," *Computers and Fluids*, Vol. 23, No. 1, 1994, pp. 103–114.

[3] Karyakin, V. E., "Computation of the Supersonic Viscous Flow over the Body of Rotation at Small Angle of Attack," *Jurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki*, Vol. 21, No. 1, 1981, pp. 150–156 (in Russian).

W. Oberkampf
Associate Editor

New Conservative Formulations of Full-Potential Equation in Streamline-Aligned Coordinates

Azat M. Latypov*
University of Windsor,
Windsor, Ontario N9B 3P4, Canada

I. Introduction

DURING the past 20 years, a number of formulations, based on the streamfunction as a coordinate (SFC) concept, i.e., writing the governing equations in a system of independent coordinates aligned with streamlines, have been used in computational fluid dynamics (CFD).[1–6] One attractive advantage is that SFC formulations permit computation of the parameters of the flow without prior grid generation in the computational domain. The governing equations play a double role of both the equations describing the motion of the media and the grid generation equations. As a result of this, both computational time and memory requirements can be reduced. Furthermore, the resulting streamline-aligned computational grid naturally conforms to the boundaries of the physical domain. These features are utilized in most of the works dealing with the SFC method in CFD especially for the solution of inverse or optimal design problems in aerodynamics.[1,3]

An essential feature of the SFC technique is the choice of the independent coordinate, which compliments the streamfunction to produce the nondegenerate system of independent coordinates. It is well known that one of the major limitations of the original von Mises approach[7] is that the resulting system of coordinates degenerates at locations where the velocity vector is normal to the axis of the transverse Cartesian coordinate. This situation is typical for the flow in the vicinity of the leading edge of an airfoil. This limits applicability of the von Mises approach to cases where one can expect in advance that the flow direction will not change significantly over the flow domain.

In the current work, a streamline-based transformation that generates an orthogonal streamwise coordinate system is used. Unlike similar formulations used in Refs. 4 and 5, the formulations developed and used in this work have an advantage in that they have a conservative form, ensuring both global and local conservation of mass and irrotationality of the velocity vector field.

II. Reformulation of Governing Equations

The assumptions that the velocity vector field is potential (irrotational) and that the flow is isentropic and isenthalpic are well established in modeling of compressible flows. Under these assumptions, the equations of momentum and energy follow from the continuity and irrotationality equations,[8] and the governing system may be

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*Research Assistant, Ph.D. Candidate, Fluid Dynamics Research Institute and Department of Mathematics and Statistics, 401 Sunset Avenue. E-mail: latypov@na-net.ornl.gov.

reduced to just one second-order equation for the velocity vector potential.

Following Ref. 8, the governing system in Cartesian coordinates (x, y) can be written as

$$\boldsymbol{w}_x + \boldsymbol{f}_y = \boldsymbol{0} \tag{1}$$

where

$$\boldsymbol{w} = \begin{bmatrix} y^\nu \rho u \\ v \end{bmatrix}, \qquad \boldsymbol{f} = \begin{bmatrix} y^\nu \rho v \\ -u \end{bmatrix}$$

where $\nu = 0$ for plane flows, $\nu = 1$ for axisymmetric flows, and density $\rho = \rho(q)$ is a known function of $q = (u^2 + v^2)^{1/2}$, as follows from the relations of constant entropy and full enthalpy.

Consider two differential relations

$$\mathrm{d}\varphi = C(\varphi)(u\,\mathrm{d}x + v\,\mathrm{d}y) \tag{2}$$

$$\mathrm{d}\psi = B(\psi) y^\nu \rho (u\,\mathrm{d}y - v\,\mathrm{d}x) \tag{3}$$

where $B(\psi)$ and $C(\varphi)$ are given positive functions, which are used to control spacing of the mesh. Note that the right-hand sides of both Eqs. (2) and (3) are full differentials because of Eq. (1). Therefore, relations (2) and (3) define a transformation from Cartesian coordinates x and y to orthogonal curvilinear coordinates φ and ψ. To write the governing equations in these new independent variables, relations (2) and (3) are inverted to obtain

$$\mathrm{d}y = uJ_1\,\mathrm{d}\psi + vJ_2\,\mathrm{d}\varphi \tag{4}$$

$$\mathrm{d}x = -vJ_1\,\mathrm{d}\psi + uJ_2\,\mathrm{d}\varphi \tag{5}$$

where $J_1 = (B(\psi)y^\nu \rho q^2)^{-1}$ and $J_2 = (C(\varphi)q^2)^{-1}$. Requiring that the right-hand sides of Eqs. (4) and (5) be full differentials results in the following formulation of the system of governing equations in the (φ, ψ) plane:

$$\boldsymbol{W}_\varphi + \boldsymbol{F}_\psi = \boldsymbol{0} \tag{6}$$

where

$$\boldsymbol{W} = \begin{bmatrix} uJ_1 \\ vJ_1 \end{bmatrix}, \qquad \boldsymbol{F} = \begin{bmatrix} -vJ_2 \\ uJ_2 \end{bmatrix}$$

A. Reduction to a Single Second-Order Equation

It follows from Eq. (4) and the first equation in Eq. (6) that

$$u/\rho q^2 = B(\psi) y^\nu y_\psi, \qquad v/q^2 = C(\varphi) y_\varphi \tag{7}$$

Substituting Eq. (7) into the second equation in Eq. (6) yields

$$(f y_\varphi)_\varphi + [(1/f) y_\psi]_\psi = 0 \tag{8}$$

where $f = J_1/J_2 = C(\varphi)/[B(\psi)y^\nu \rho]$. Notice that Eq. (8) is the second of Eqs. (6), written in terms of y and its derivatives, and that the first of Eqs. (6) is also satisfied because of relations (7). Therefore, Eq. (8) can be taken as a governing equation in place of system (6).

If the unknown function $y = y(\varphi, \psi)$ is found, the primitive variables u and v can be calculated from Eq. (7), because in an isentropic and isenthalpic flow ρ is a known function of q and, therefore, Eq. (7) becomes a system of two algebraic equations for the unknowns u and v.

However, as direct calculation shows,

$$\frac{\partial(u/\rho q^2, v/q^2)}{\partial(u, v)} = -\frac{1}{\rho q^4}\left(1 - \frac{u^2}{a^2}\right)$$

where a is the speed of sound. Therefore, the algebraic system of Eqs. (7) degenerates on limiting lines[8] of the flow where $u^2/a^2 = 1$. This implies that special consideration, allowing the choice of the correct root in the algebraic system (7), needs to be used when computing transonic flows with limiting lines.

B. System of Two Second-Order Equations

In this section, an alternative formulation, which makes it possible to avoid the problem of nonuniqueness of the solution to system (7), is derived.

Using Eqs. (4) and (5), the fluxes $\boldsymbol{W}$ and $\boldsymbol{F}$ can be written in terms of the partial derivatives of the functions $x = x(\varphi, \psi)$ and $y = y(\varphi, \psi)$:

$$\boldsymbol{W} = \begin{bmatrix} f x_\varphi \\ f y_\varphi \end{bmatrix}, \qquad \boldsymbol{F} = \begin{bmatrix} x_\psi / f \\ y_\psi / f \end{bmatrix}$$

and u and v can be determined from

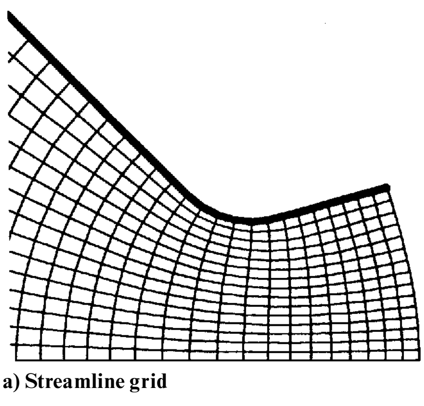
$$u/q^2 = C(\varphi) x_\varphi, \qquad v/q^2 = C(\varphi) y_\varphi \tag{9}$$

Unlike system (7), system (9) can always be resolved uniquely with respect to u and v. Equations (6) then become a system of two coupled nonlinear second-order equations for x and y.

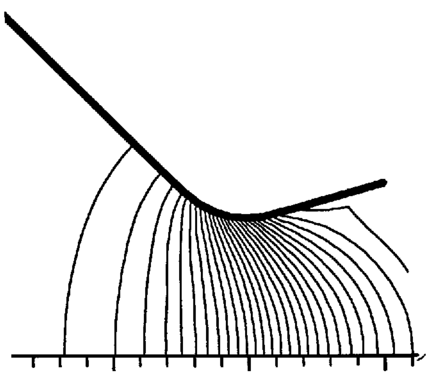
III. Numerical Results

Figure 1 shows the results of application of the formulation developed in Sec. II.A to the calculation of the transonic potential gas flow in a converging–diverging nozzle. Equation (8) has been approximated using central differences and artificial compressibility applied in transonic regions. The resulting set of finite difference equations has been solved using approximate factorization iterative technique. To deal with nonuniqueness of the solution of the algebraic system (7), a finite difference approximation of the differential equation for Mach number, which follows from Eq. (6), was used at the transonic nodes of the grid to find u and v from known distribution of $y = y(\varphi, \psi)$ on the current iteration, as suggested in Ref. 6.

Figure 2 shows results obtained using the formulation described in Sec. II.B. The subsonic flow through a bumpy axisymmetric channel

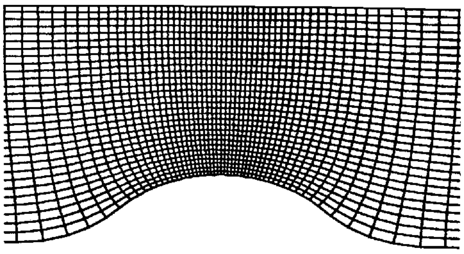


a) Streamline grid

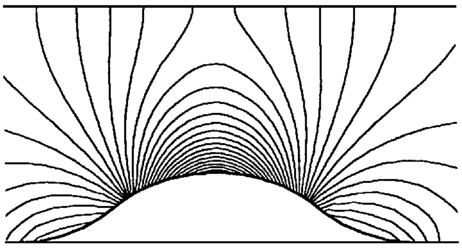


b) Pressure contours

Fig. 1 Transonic flow in a nozzle.



a) Streamline grid



b) Mach number contours

Fig. 2 Subsonic flowfield in the axisymmetric bumpy channel.

was calculated using central difference approximation and point Gauss–Seidel iterative algorithm with multigrid acceleration.

The implementation details of the numerical algorithms used to compute the flows presented in Figs. 1 and 2, as well as a generalization of the coordinate transformation Eqs. (2) and (3) and applications to steady supersonic vortical flow calculations, can be found in Ref. 9.

IV. Conclusions

The new formulations for compressible potential flow equations presented here utilize an orthogonal streamline-aligned system of independent natural, body-fitting coordinates. An advantage of the presented formulations over the similar previously known ones[1,3–5] is that the formulations developed in this work are expressed as meaningful differential conservation laws, which allows use of them to obtain conservative finite volume approximations.

References

[1]Barron, R. M., An, C.-F., and Zhang, S., "Survey of the Streamfunction-as-a-Coordinate Method in CFD," *Proceedings of the Inaugural Conference of the CFD Society of Canada*, CFD Society of Canada, Montreal, PQ, Canada, 1993, pp. 325–336.

[2]Liou, M.-S., "An Extended Lagrangian Method," *Journal of Computational Physics*, Vol. 118, No. 2, 1995, pp. 294–309.

[3]Dulikravich, G. S., "A Stream-Function-Coordinate (SFC) Concept in Aerodynamic Shape Design," AGARD-R-780, Brussels, Belgium, 1990, pp. 6.1–6.6.

[4]Stanitz, J. D., "Design of Two-Dimensional Channels with Prescribed Velocity Distributions Along the Channel Walls," NACA TN 2595, May 1952.

[5]Osipov, I. L., Shipilin, A. V., and Shulishnina, N. P., "A Numerical Method for Calculation of Gas Flows in Channels and Nozzles in Direct, Inverse and Combined Modes," *Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki*, Vol. 27, No. 10, 1987, pp. 1563–1572 (in Russian).

[6]Latypov, A. M., and Shipilin, A. V., "On an Iterational Method for Calculating Sub- and Transonic Internal Flows," *USSR Computational Mathematics and Mathematical Physics*, Vol. 31, No. 5, 1991, pp. 92–99.

[7]von Mises, R., "Bemerkungen zur Hydrodynamik," *ZAMM*, Vol. 7, 1927, p. 425.

[8]Ovsyannikov, L. V., *Lectures on Fundamentals of Gas Dynamics*, Nauka, Moscow, 1981, Chap. 4 (in Russian).

[9]Latypov, A. M., *Numerical Solution of Steady Euler Equations in Streamline-Aligned Orthogonal Coordinates*, IMA Preprint Series No. 1182, Inst. for Mathematics and Its Applications, Univ. of Minnesota, Minneapolis, MN, Nov. 1993, pp. 1–33.

D. S. McRae
Associate Editor

Vorticity Jump in Surface Coordinates Across a Shock in Nonsteady Flow

Z. U. A. Warsi*
Mississippi State University,
Mississippi State, Mississippi 39762

Introduction

RECENT advances in various areas of computational fluid dynamics (CFD) have prompted the use of the formulas developed by Hayes[1] for the calculation of vorticity jump across a shock. Very recently an interesting discussion of Hayes' work by Isom and Kalkhoran[2] and Emanuel[3] has appeared, which further tends to establish the correctness of the work reported in Ref. 1. The simplicity and elegance of the formulas derived by Hayes are due to his choice of the normal coordinate as straight lines in both the steady and nonsteady flow situations. Obviously curved normal coordinates can be chosen equally but their introduction will involve unnecessary algebra with no change in the jump condition results.

The purpose of this Note is to obtain Hayes' formulas in the coordinates attached to an arbitrarily moving and deforming shock surface that occurs in nonsteady flows. The formulas given here can be incorporated in a flow solver that uses a coordinate system attached to a moving and deforming shock surface. Further, the essential geometrical properties of the shock surface are shown to depend on quantities that are directly computable.

In the present problem, the geometry of the shock surface changes with time and, therefore, a coordinate generator has to be used for the generation of surface coordinates and also for the calculation of the elements of the surface curvature tensor. It seems that the surface coordinate generator as developed by Warsi[4] (refer also to Ref. 5, p. 649) is most suitable for the present purpose inasmuch as the coordinate generation equations in Ref. 4 explicitly depend on the geometrical properties of the surface, and they also satisfy other differential-geometric properties.[6]

Analysis

Let x^i, $i = 1, 2, 3$, be a general time-dependent coordinate system such that x^α, $(\alpha = 1, 2)$, is a general coordinate system in the shock surface and $x^3 = n$ the actual normal distance from the shock surface at any given time. Thus, the coordinates form a local parallel surface coordinate system at a given instant. Let $\boldsymbol{r} = (x, y, z)$ be a point on the shock surface, then $\boldsymbol{a}_\alpha = \partial \boldsymbol{r}/\partial x^\alpha$, $\alpha = (1, 2)$, are the covariant base vectors in the shock surface, and the grad or nebla operator (repeated indices implying a sum) is

$$\text{grad} = \boldsymbol{a}^\alpha \frac{\partial}{\partial x^\alpha} + \boldsymbol{n}\frac{\partial}{\partial n} \tag{1a}$$

where $\boldsymbol{n}$ is the unit normal vector at the surface drawn along the direction of increasing n. Thus,

$$\begin{aligned} \boldsymbol{a}^\alpha &= \text{grad}\, x^\alpha, \qquad & \boldsymbol{a}^3 &= \boldsymbol{a}_3 = \boldsymbol{n} \\ g^{13} &= g_{23} = 0, \qquad & g_{33} &= g^{33} = 1 \end{aligned} \tag{1b}$$

(Ref. 5, p. 579), where g_{ij} and g^{ij} are the covariant and the contravariant metric coefficients, respectively. In the ensuing analysis, we have adopted the nondyadic operation for the gradient of vectors and divergence of tensors (Ref. 5, p. 602), i.e.,

$$\text{grad}\, \boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial x^\alpha}\boldsymbol{a}^\alpha + \frac{\partial \boldsymbol{u}}{\partial n}\boldsymbol{n} \tag{1c}$$

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*Professor, Department of Aerospace Engineering. Member AIAA.